Pergamon

J. Appl. Maths Mechs, Vol. 59, No. 2, pp. 249–254, 1995

0021–8928/95 $24.00+0.00

0021–8928(95)00039-9

THE SLIP LINE AT THE END OF A PUNCH IMPRESSED INTO A HALF-PLANE†

V. M. ALEKSANDROV and L. A. KIPNIS

Moscow, Uman

(*Received* 26 *January* 1994)

Under the conditions of the plane static problem (plane strain) the problem of determining the initial plasticity zone near the end of a rigid punch impressed into an elastoplastic half-plane is considered. The plasticity zone is modelled by a straight slip line emerging from the end of the punch. Taking the shortness of the slip line into account the corresponding boundary-value problem is formulated for a half-plane with a non-symmetric rectilinear cut which emerges at its boundary. An exact solution of the Wiener–Hopf functional equation is constructed and used to derive an equation for determining the length of the slip line. The equation is shown to be solvable. The direction in which the slip line develops is established from the condition for its length to be a maximum. Expressions for the length of the slip line and its angle of inclination to the boundary are derived for various values of the parameters.

1. Under plane strain conditions we consider the static problem of the impression of a rigid punch of angle α at its vertex O into an elastoplastic half-plane (Fig. 1). The parallel sides of the punch are perpendicular to the boundary of the half-plane. It is assumed that under the action of the impressing force P the punch will penetrate perpendicularly into the boundary of the half-plane. There is no friction between the punch and the surface of the elastoplastic body. Outside the contact line the body surface is stress free.

However small the impressing force may be, a plasticity zone appears near the stress-concentrating point O. We shall only study the initial stage of its development, on the assumption that the impressing force is sufficiently small. The size of the plasticity zone will then be small compared with the length of the contact line and the size of the punch. Following the now widely-used and experimentally confirmed localization hypothesis [1–3], we will model the initial plasticity zone by a straight slip line emerging from the point O. Only the tangential displacement is discontinuous along the slip line, and the shear stress is equal to $\pm\tau_s$ (where τ_s is the shear yield point). The sign in front of τ_s is decided when specific problems are formulated. In this case, in the corresponding elastic problem

$$r_0 \to 0, \quad 0<\theta_0<\pi, \quad \tau_{r_0\theta_0} \sim \frac{Kf(\theta_0)}{4\sqrt{2\pi r_0}}, \quad f(\theta_0)=\sin\frac{\theta_0}{2}+\sin\frac{3\theta_0}{2}$$

(where $\tau_{r_0\theta_0}$ is the stress and K is a negative constant).

Since $f(\theta_0) > 0$, for every θ_0 near the point O we have $\tau_{r_0\theta_0} < 0$. Hence a minus sign should be chosen in front of τ_s.

In view of the fact that $f(\theta_0)$ reaches its maximum value at $\theta_0 = 2\arccos(\sqrt{2}/\sqrt{3})$, the slip line should be expected to develop at an angle of approximately 109° to the stress-free $\theta_0 = \pi$ part of the boundary. This conclusion will be verified below by analysing the solution of the boundary-value problem.

Because the slip line is short compared with the length of the contact line and the size of the punch, and because in what follows we shall use information on the stress–strain state only near the point O, we shall use as the solution of this problem the solution of the corresponding problem for a semi-infinite punch. Here the curve OO_1, that is smooth by assumption, can be replaced by any other smooth curve making an angle α with the line OO_2.

The boundary conditions are (see Fig. 1)

$$\theta=\beta, \quad \sigma_\theta=\tau_{r\theta}=0; \quad \theta=\beta-\pi, \quad \tau_{r\theta}=0, \quad u_\theta=A\ln\frac{r+L}{L}+\text{const}$$

$$\theta=0, \quad \langle\sigma_\theta\rangle=\langle\tau_{r\theta}\rangle=0; \quad \langle u_\theta\rangle=0 \tag{1.1}$$

†*Prikl. Mat. Mekh.* Vol. 59, No. 2, pp. 266–271, 1995.

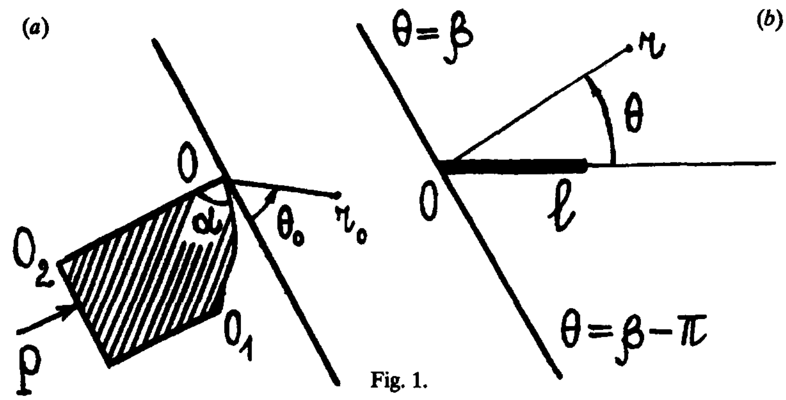

Fig. 1.

$$\theta=0,\quad r<l,\quad \tau_{r\theta}=-\tau_s;\quad \theta=0,\quad r>l,\quad \langle u_r\rangle=0 \tag{1.2}$$

$$\theta=0,\quad r\to l-0,\quad \left\langle\frac{\partial u_r}{\partial r}\right\rangle\sim-\frac{4(1-\nu^2)}{E}\frac{K_{\rm II}}{\sqrt{2\pi(l-r)}} \tag{1.3}$$

$$\theta=0,\quad r\to l+0,\quad \tau_{r\theta}\sim\frac{K_{\rm II}}{\sqrt{2\pi(r-l)}}$$

$$\int_0^\infty \sigma_\theta(r,\beta-\pi)dr=-P \tag{1.4}$$

Here σ_θ and $\tau_{r\theta}$ are stresses, u_θ and u_r are displacements, $\langle n\rangle$ is the jump in n at $\theta = 0$, E is Young's modulus, ν is Poisson's ratio, $k_{\rm II}$ is the stress intensity factor at the end of the slip line, to be determined, A is a negative constant whose value is chosen when solving the problem so that condition (1.4) is satisfied, and $L = -A$ tg α.

It is required to find the length l of the slip line and its angle of inclination β to the boundary of the half-plane.

These quantities are determined below using the scheme employed in [3]. The length of the slip line is found from the condition that the stress intensity factor vanishes at its end. It is a function of the angle of inclination of the slip line of the boundary of the half-plane which is a free parameter of the problem. This angle, which gives the initial direction of development of the slip line, is found by choosing the value of the free parameter which maximizes its length.

2. Applying a Mellin transformation [4]

$$m^*(p)=\int_0^\infty m(r)r^p dr$$

to the equilibrium equations, the strain compatibility condition and Hooke's law, and using (1.1) and (1.2), we obtain the following relations for the Mellin transforms

$$\sigma_\theta^*(p,\beta,-\pi)=g^{-1}(p)\{-4[\Delta_1(p)\Delta_4(p)+\Delta_2(p)\Delta_3(p)]u(p)+$$
$$+2[2\Delta_3(p)\sin(p+1)(\pi-\beta)-2d(p)-(p-1)\Delta_1(p)\sin p(\pi-\beta)\sin\beta]\tau_{r\theta}^*(p,0)\} \tag{2.1}$$

$$\tau_{r\theta}^*(p,0)=-\frac{l^{p+1}g(p)\Phi^-(p)+4u(p)(p+1)\sin p\pi\sin p\beta\sin\beta}{\sin 2p\pi} \tag{2.2}$$

$$g(p)=\Delta_1(p)\Delta_2(p)+2\Delta_3(p)[\cos 2p(\pi-\beta)-\cos 2\beta]$$

$$\Delta_1(p)=\sin 2p\beta+p\sin 2\beta,\quad \Delta_2(p)=\sin 2p(\pi-\beta)-p\sin 2\beta$$

$$\Delta_3(p)=\sin^2 p\beta-p^2\sin^2\beta,\quad \Delta_4(p)=\sin^2 p(\pi-\beta)-p^2\sin^2\beta$$

$$d(p)=(\sin^2 p\beta-p\sin^2\beta)[p\cos p(\pi-\beta)\sin\beta-\sin p(\pi-\beta)\cos\beta]$$

$$u(p)=\frac{\pi AE}{4(1-\nu^2)}\frac{L^p}{\sin p\pi}$$

$$\Phi^-(p)=-\frac{E}{4(1-\nu^2)}\int_0^1\left\langle\frac{\partial u_r}{\partial r}\right\rangle_{r=\rho l}\rho^p d\rho$$

($-\varepsilon_1 < \operatorname{Re} p < \varepsilon_2$ where ε_1 and ε_2 are sufficiently small positive numbers). Putting $p = 0$ into (2.1) and satisfying (1.4), we find

$$A=-\frac{1-\nu^2}{\pi}a(\beta)\frac{P}{E}$$

$$a(\beta)=\frac{(2\pi-2\beta-\sin 2\beta)(2\beta+\sin 2\beta)+4(\beta^2-\sin^2\beta)\sin^2\beta}{\pi(2\beta+\sin 2\beta)-2(\beta\cos\beta+\sin\beta)^2}$$

Using (1.2) and (2.2) we arrive at the Wiener–Hopf functional equation

$$\Phi^+(p)+a_1(\beta)\frac{P}{l}U(p)-\frac{\tau_s}{p+1}=-\operatorname{tg}p\pi G(p)\Phi^-(p) \tag{2.3}$$

$$(-\varepsilon_1<\operatorname{Re}p<\varepsilon_2)$$

$$G(p)=\frac{g(p)}{2\sin^2 p\pi},\quad U(p)=\frac{(p+1)\sin p\beta}{\sin 2p\pi}\lambda^{-p},\quad \lambda=\frac{l}{L}$$

$$\Phi^+(p)=\int_1^\infty\tau_{r\theta}(\rho l,0)\rho^p d\rho,\quad a_1(\beta)=a(\beta)\sin\beta$$

Using the factorization

$$G(p)=\frac{G^+(p)}{G^-(p)}(\operatorname{Re}p=0),\quad \exp\left[\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln G(z)}{z-p}dz\right]=\begin{cases}G^+(p), & \operatorname{Re}p<0\\ G^-(p), & \operatorname{Re}p>0\end{cases} \tag{2.4}$$

$$p\operatorname{ctg}p\pi=K^+(p)K^-(p),\quad K^\pm(p)=\frac{\Gamma(1\mp p)}{\Gamma(\tfrac{1}{2}\mp p)}$$

(where $\Gamma(z)$ is the Gamma function), we rewrite Eq. (2.3) as

$$\frac{K^+(p)\Phi^+(p)}{G^+(p)}+a_1(\beta)\frac{P}{l}\frac{K^+(p)U(p)}{G^+(p)}-\tau_s\frac{K^+(p)}{(p+1)G^+(p)}=-\frac{p\Phi^-(p)}{K^-(p)G^-(p)} \tag{2.5}$$

$$(\operatorname{Re}p=0)$$

Since

$$\frac{K^+(p)U(p)}{G^+(p)}=U^+(p)-U^-(p)\quad(\operatorname{Re}p=0)$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{K^+(z)U(z)}{G^+(z)(z-p)}dz=\begin{cases}U^+(p), & \operatorname{Re}p<0\\ U^-(p), & \operatorname{Re}p>0\end{cases}$$

$$\frac{K^+(p)}{(p+1)G^+(p)}=\frac{1}{p+1}\left[\frac{K^+(p)}{G^+(p)}-\frac{K^+(-1)}{G^+(-1)}\right]+\frac{K^+(-1)}{G^+(-1)(p+1)}$$

$$(\operatorname{Re}p=0)$$

from (2.5) we obtain

$$\frac{K^+(p)\Phi^+(p)}{G^+(p)}+a_1(\beta)\frac{P}{l}U^+(p)-\frac{\tau_s}{p+1}\left[\frac{K^+(p)}{G^+(p)}-\frac{K^+(-1)}{G^+(-1)}\right]=$$
$$=-\frac{p\Phi^-(p)}{K^-(p)G^-(p)}+a_1(\beta)\frac{P}{l}U^-(p)+\frac{K^+(-1)\tau_s}{G^+(-1)(p+1)} \tag{2.6}$$
$$(\operatorname{Re} p=0)$$

The function on the left-hand side of (2.6) is analytic in the Re $p<0$ half-plane, while the function on the right is analytic in the Re $p>0$ half-plane. Using the principle of analytic continuation these functions are equal to one and the same function which is analytic over the whole of the p plane.

Starting from (1.3), by a theorem of Abel type [5] we find

$$\Phi^+(p)\sim\frac{k_{\mathrm{II}}}{\sqrt{-2pl}},\quad \Phi^-(p)\sim-\frac{k_{\mathrm{II}}}{\sqrt{2pl}}\quad (p\to\infty) \tag{2.7}$$

From (2.4) and (2.7) it follows that the functions on the left- and right-hand sides of (2.6) tend to $k_{\mathrm{II}}/\sqrt{(2l)}$ as $p\to\infty$ in the half-planes Re $p<0$ and Re $p>0$, respectively. By Liouville's theorem the unique analytic function is identically equal to $k_{\mathrm{II}}/\sqrt{(2l)}$ over the whole of the p plane.

Putting $p=0$ on the right-hand side of (2.6), we conclude that the unique analytic function is equal to $C=a_1(\beta)P/lU^-(0)+K^+(-1)/G^+(-1)\tau_s$.

The solution of Eq. (2.3) therefore has the form

$$\Phi^+(p)=\frac{G^+(p)}{K^+(p)}\left\{C-a_1(\beta)\frac{P}{l}U^+(p)+\frac{\tau_s}{p+1}\left[\frac{K^+(p)}{G^+(p)}-\frac{K^+(-1)}{G^+(-1)}\right]\right\}$$
$$(\operatorname{Re} p<0)$$
$$\Phi^-(p)=-\frac{K^-(p)G^-(p)}{p}\left[C-a_1(\beta)\frac{P}{l}U^-(p)-\frac{K^+(-1)\tau_s}{G^+(-1)(p+1)}\right]$$
$$(\operatorname{Re} p>0)$$

and the stress intensity factor at the end of the slip line is given by

$$k_{\mathrm{II}}=\left[a_1(\beta)\frac{P}{l}U^-(0)+\frac{2}{\sqrt{\pi}G^+(-1)}\tau_s\right]\sqrt{2l} \tag{2.8}$$

3. Using (2.8) we obtain the following equation for determining the length of the slip line

$$\varphi(\lambda)=0,\quad \varphi(\lambda)=U^-(0)+QM\lambda,\quad M=\frac{\tau_s\operatorname{tg}\alpha}{E},\quad Q=\frac{2(1-\nu^2)}{\pi\sqrt{\pi}G^+(-1)\sin\beta} \tag{3.1}$$

From the Sokhotskii formula

$$U^-(0)=Z+I,\quad I=\lim_{\varepsilon\to 0}I_\varepsilon$$
$$Z=-\frac{1}{2}\frac{K^+(0)U(0)}{G^+(0)}=-\frac{\beta}{2}\{2\pi[(2\pi-2\beta-\sin 2\beta)(2\beta+\sin 2\beta)+4(\beta^2-\sin^2\beta)\sin^2\beta]\}^{-1/2}$$
$$I_\varepsilon=\frac{1}{2\pi i}\left[\int_{-i\infty}^{-i\varepsilon}\frac{K^+(z)U(z)}{G^+(z)z}dz+\int_{i\varepsilon}^{i\infty}\frac{K^+(z)U(z)}{G^+(z)z}dz\right]$$

We will study the behaviour of $\varphi(\lambda)$ near the point $\lambda=0$. Consider the integral

$$I_\gamma=\frac{1}{2\pi i}\int_\gamma\frac{K^+(z)U(z)}{G^+(z)z}dz$$

The contour γ consists of the semi-intervals $]-i\infty,-i\varepsilon]$, $[i\varepsilon;i\infty[$ and a semicircle of radius ε centred at

$z = 0$ in the half-plane $\operatorname{Re} z < 0$. The orientation of the contour coincides with the direction of the imaginary axis.

This integral is equal to the sum Σ of the residues of the integrand at those of its poles that lie in the $\operatorname{Re} z < 0$ half-plane. On the other hand, it is equal to $I\varepsilon + I_0$, where I_0 is the integral over the above semicircle multiplied by $1/(2\pi i)$. Hence $I_\varepsilon \to I$, $I_0 \to Z$ as $\varepsilon \to 0$, and $U^-(0) = \Sigma$.

From among the poles of the integrand which lie in the $\operatorname{Re} z < 0$ half-plane and correspond to the integral I_γ, the one closest to the imaginary axis is the point $z = -{}^1/_2$. Hence

$$\varphi(\lambda) \sim -\frac{\sin\beta/2}{4\sqrt{\pi}G^+(-\frac{1}{2})}\sqrt{\lambda} \quad (\lambda \to 0) \tag{3.2}$$

It follows from (3.2) that the function $\varphi(\lambda)$, which tends to zero as $\lambda \to 0$, is negative near the point $\lambda = 0$.

As a result of appropriate transformations $\varphi(\lambda)$ acquires the form

$$\varphi(\lambda)\int_0^\infty [F_1(t)\cos t\ln\lambda + F_2(t)\sin t\ln\lambda]dt + QM\lambda$$

$$F_1(t) = \frac{1}{\pi}s(t)\left[f_1(t) + \frac{f_2(t)}{t}\right], \quad F_2(t) = \frac{1}{\pi}\left[s(t)f_2(t) - \frac{2Z + s(t)f_1(t)}{t}\right]$$

$$f_1(t) = f_0(t)[i_+(t)\cos i_0(t) - i_-(t)\sin i_0(t)], \quad f_2(t) = f_0(t)[i_-(t)\cos i_0(t) + i_+(t)\sin i_0(t)]$$

$$f_0(t) = [G(it)]^{-\frac{1}{2}}[i_3^2(t) + i_4^2(t)]^{-1}$$

$$i_+(t) = i_1(t)i_3(t) + i_2(t)i_4(t), \quad i_-(t) = i_1(t)i_4(t) - i_2(t)i_3(t)$$

$$i_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\ln G(i\upsilon)}{\upsilon - t}d\upsilon, \quad i_1(t) = \int_0^\infty e^{-\upsilon}\cos t\ln\upsilon d\upsilon, \quad i_2(t) = \int_0^\infty e^{-\upsilon}\sin t\ln\upsilon d\upsilon$$

$$i_3(t) = \int_0^\infty e^{-\upsilon}\upsilon^{-\frac{1}{2}}\cos t\ln\upsilon d\upsilon, \quad i_4(t) = \int_0^\infty e^{-\upsilon}\upsilon^{-\frac{1}{2}}\sin t\ln\upsilon d\upsilon, \quad s(t) = \frac{\operatorname{sh} t\beta}{\operatorname{sh} 2t\pi}$$

Let δ be a sufficiently small positive number and

$$\tau_s > \frac{E}{Q\delta\operatorname{tg}\alpha}\max\left[\int_0^\infty F(t)dt, \ \int_0^\infty \tilde{F}(t)dt\right]$$

$$F(t) = |F_1(t)| + |F_2(t)|, \quad \tilde{F}(t) = t|F_1(t)| + \frac{s(t)}{\pi}(|f_1(t)| + t|f_2(t)|)$$

Then $\varphi(\delta) > 0$, $\varphi'(\lambda) > 0$ or $\lambda \geqslant \delta$, and hence $\varphi(\lambda) > 0$ when $\lambda \geqslant \delta$. Using the negativity of $\varphi(\lambda)$, demonstrated above near the point $\lambda = 0$, we conclude that Eq. (3.1) is solvable in the interval $]0; \delta[$, whereas when $\lambda \geqslant \delta$ there is no solution. Calculations for a large number of values of the parameters β, ν and M over a wide range of variation for the latter show that the solution is unique.

If $\lambda = s(\beta, \nu, M)$ is a solution of Eq. (3.1), then

$$l = D\frac{P}{l}, \quad D = \frac{1 - \nu^2}{\pi}\operatorname{tg}\alpha a(\beta)s(\beta, \nu, M)$$

The value of β_* giving the largest value of the function $\chi(\beta) = as$ determines the angle of inclination of the slip line to the boundary of the half-plane.

Table 1 gives some values for D when $\nu = 0.333$; for the values of τ_s given there the value of β_* was 107° when $\alpha = 15°$, 108° when $\alpha = 45°$ and $\alpha = 80°$, and 109° when $\alpha = 89°$.

For fixed α, ν, E, P as τ_s decreases and for fixed ν, τ_s, E, P as α decreases the length of the slip line increases.

The results of the calculations enable us to assume that the angle β_* is approximately equal to 107–109°, which agrees with what was said in Section 1 on the direction of propagation of the slip line.

We also note that for the contact stress $\sigma_\theta(r, \beta - \pi)$ the characteristic square-root singularity at the point $r = 0$ (the start of the slip line) is not present.

Table 1

$\alpha°$	$10^3\tau_s/E = 1$	5	7	10
15	0,2114	0,2069	0,2042	0,2007
45	0.1496	0,1443	0,1420	0,1398
80	0,0801	0,0756	0,0731	0,0711
89	0,0571	0,0537	0,0512	0,0492

REFERENCES

1. LEONOV M. Ya., VITVITSKII P. M. and YAREMA S. Ya., Plasticity strips in the stretching of plates with crack-shaped concentrators. *Dokl. Akad. Nauk SSSR* **148**, 3, 541–544, 1963.
2. VITVITSKII P. M., PANASYUK V. V. and YAREMA S. Ya., Plastic deformations in the neighbourhood of cracks and criteria of fracture (review). *Problemy Prochnosti* 2, 3–18, 1973.
3. CHEREPANOV G. P., Plastic lines of discontinuity at the tip of a crack. *Prikl. Mat. Mekh.* **50**, 4, 720–728, 1976.
4. UFLYAND Ya. S., *Integral Transformations in Problems of the Theory of Elasticity*. Nauka, Moscow, 1967.
5. NOBLE B., *The Use of the Wiener–Hopf Method to Solve Partial Differential Equations*. IIL, Moscow, 1962.

Translated by R.L.Z.